

FRACTAL DIMENSIONS OF SUBFRACTALS INDUCED BY SOFIC SUBSHIFTS

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ABSTRACT. In this paper, we will consider subfractals of hyperbolic iterated function systems which satisfy the open set condition. The subfractals will consist of points associated with infinite strings from a subshift of finite type or sofic subshift on the symbolic space. We find that the zeros of the lower and upper topological pressure functions are lower and upper bounds, respectively, for the Hausdorff, packing, lower and upper box dimensions of the subfractal.

1. INTRODUCTION

One area of interest in fractal geometry is the study of properties which distinguish two distinct fractals. In particular, fractal dimensions, such as Hausdorff, box, and packing dimensions, have proven to be useful properties that in a sense, extend our usual notion of topological dimension. Numerous results exist for calculating the exact value of fractal dimensions of certain fractals of IFS type, such as self-similar IFSs [4,5,7], or finding bounds for the fractal dimensions for hyperbolic IFSs [2,8]. In this paper, we will focus on specific subsets of fractals of IFS type, namely, subfractals.

Clearly, not every subset of a fractal exhibits fractal-like properties; hence, one must provide a precise definition of a subfractal to produce a genuinely different fractal. For example, a subset of an IFS fractal may be a contracted copy of the entire fractal, which inherits most of the important properties (including fractal dimensions) from the whole fractal.

In this paper, we have chosen to identify a subfractal of an IFS type fractal by only considering those points associated with a subshift on the associated symbolic space. Unless stated otherwise, a subfractal will refer to a subshift-type subfractal for the remainder of this paper. If a subshift of finite type (SFT) or sofic subshift is chosen, we find that if the subshift does not have full Hausdorff dimension, then the Hausdorff dimension of the subfractal is strictly less than the Hausdorff dimension of the original fractal.

Let \mathcal{A} denote a finite alphabet, X denote the full shift, and $Y \subset X$ be a subshift. If Y is a SFT, then there exists a matrix A consisting only of 0's and 1's which is associated with the subshift. The entries of A are determined by finite strings in the subshift which are either allowed or not allowed to appear. We will use the notation $Y = X_A$ for the subshift associated with A .

Let $\mathcal{K} \subset \mathbb{R}^n$ be a compact subset and $(\mathcal{K}; f_1, \dots, f_n)$ be an IFS with $f_i : \mathcal{K} \rightarrow \mathcal{K}$ for $1 \leq i \leq n$. The IFS $(\mathcal{K}; f_1, \dots, f_n)$ is *hyperbolic* if for all $x, y \in \mathcal{K}$, there exists some constant c such that $d(f_i(x), f_i(y)) \leq cd(x, y)$, for all $1 \leq i \leq n$. Let F denote the attractor of this hyperbolic IFS (HIFS), i.e. F is a non-empty, closed set with

$f_i(F) \subset F$ for $1 \leq i \leq n$ and the smallest such set that satisfies these properties. By [5], we know that an attractor exists for any HIFS. Recall that an IFS satisfies the open set condition (OSC) if there exists a nonempty open subset $U \subset \mathcal{K}$ such that $f_i(U) \subset U$ and $f_i(U) \cap f_j(U) = \emptyset$ for $i \neq j$ and all $1 \leq i, j \leq n$.

Now, let \mathcal{F}_{X_A} be the collection of all points from the full fractal which are associated with a sequence in X_A , i.e. $x \in \mathcal{F}_{X_A}$ if there exists some $\omega = \omega_1\omega_2 \dots \in X_A$ such that $x = \lim_{k \rightarrow \infty} [f_{\omega_k} \circ f_{\omega_{k-1}} \circ \dots \circ f_{\omega_1}(y)]$, $y \in \mathcal{K}$. Let $\rho(A)$ denote the spectral radius of a square matrix A . We prove the following:

Theorem (Main Theorem A). *For a compact subset $\mathcal{K} \subset \mathbb{R}^n$, let $\{\mathcal{K}; f_i : 1 \leq i \leq m\}$ be an HIFS which satisfies the OSC. Let $0 < c_i \leq \bar{c}_i < 1$ denote the constants such that $c_i d(x, y) \leq d(f_i(x), f_i(y)) \leq \bar{c}_i d(x, y)$ for all $1 \leq i \leq m$. Let X_A be an SFT with an associated irreducible square $(0, 1)$ -matrix A . Let \mathcal{F}_{X_A} denote the subfractal associated with the subshift. Then,*

$$h \leq \dim_H(\mathcal{F}_{X_A}) \leq H \text{ and } h \leq \overline{\dim}_B(\mathcal{F}_{X_A}) \leq H,$$

where $\rho(AS^{(h)}) = 1 = \rho(A\bar{S}^{(H)})$, S and \bar{S} are the corresponding diagonal matrices with appropriate constants c_i and \bar{c}_i on the diagonal and 0's elsewhere, $(S^{(h)})_{ij} = [S_{ij}]^h$ for all $1 \leq i, j \leq N$ and $\bar{S}^{(H)}$ is defined similarly.

Next, we turn our attention to a broader class of subshifts, the sofic subshifts. The class of sofic subshifts not only contains all SFTs but also all factors of SFTs. A common example of a sofic subshift which is not an SFT is the Golden Mean Shift, which has forbidden word list $F = \{101, 10001, \dots, 10^{2k+1}1, \dots\}$ on the alphabet $\mathcal{A} = \{0, 1\}$.

As in the case of subfractals induced by SFTs, a subfractal induced by a sofic subshift can be represented by a matrix $A_{\mathcal{G}}$; however, the entries of $A_{\mathcal{G}}$ consist of sums of contractive factors associated with finite, allowable strings from the subshift determined by an underlying labeled graph \mathcal{G} . Hence, we must alter the techniques we used for SFTs to compensate for the differences in the matrices associated with the subshifts. Let \mathcal{L} denote the labeling with the lower contractive bounds and $\bar{\mathcal{L}}$ denote the labeling with the upper contractive bounds. See Section 5 for details on this labeling. We prove the following:

Theorem (Main Theorem B). *For a compact subset $\mathcal{K} \subset \mathbb{R}^n$, let $\{\mathcal{K}; f_i : 1 \leq i \leq m\}$ be an HIFS which satisfies the OSC. Let $0 < c_i \leq \bar{c}_i < 1$ denote the constants such that $c_i d(x, y) \leq d(f_i(x), f_i(y)) \leq \bar{c}_i d(x, y)$ for all $1 \leq i \leq m$. Let $X_{\mathcal{G}}$ be a sofic subshift with irreducible matrices $A = (a_{ij})_{1 \leq i, j \leq k}$ and $\bar{A} = (\bar{a}_{ij})_{1 \leq i, j \leq k}$ with $a_{ij} = \sum_{e_{ij}} \mathcal{L}(e_{ij})$ and $\bar{a}_{ij} = \sum_{e_{ij}} \bar{\mathcal{L}}(e_{ij})$. Then,*

$$h \leq \dim_H(\mathcal{F}_{X_{\mathcal{G}}}) \leq H \text{ and } h \leq \overline{\dim}_B(\mathcal{F}_{X_{\mathcal{G}}}) \leq H,$$

where $\rho(A_h) = 1 = \rho(A_H)$, and A_h, A_H have entries $a_{ij}^{(h)} = \sum_{e_{ij}} \mathcal{L}(e_{ij})^h$, $a_{ij}^{(H)} = \sum_{e_{ij}} \bar{\mathcal{L}}(e_{ij})^H$, respectively.

Remark 1. *Theorem A will be split into Theorems 4.6 and 4.7 (for Hausdorff dimension bounds and upper box dimension bounds, respectively) in the case where A is an irreducible matrix. Similarly, the case in which matrix $A_{\mathcal{G}}$ from Theorem B is irreducible will be presented as Theorem 5.2. Theorem 6.3 will extend the results*

for Hausdorff dimension of subfractals defined by either a SFT or sofic subshift with a reducible matrix.

These results generalize previously proven results, including Theorems 1.1 and 1.2 below, which analyze different types of subfractals [2,8]. Following the notation and terminology in [8], we say that A is *primitive* if there exists some integer N such that $(A^N)_{ij} > 0$ for all $1 \leq i, j \leq n$, where $(A^N)_{ij}$ denotes the ij -entry of A^N . A sequence of integers $(i_l)_{l \geq 1}$, where $i_l \in \{1, \dots, n\}$, is said to be *admissible* if $(A)_{i_l, i_{l+1}} \neq 0$ for all $l \geq 1$. Let F_A denote the collection of all points in F which are associated with an admissible sequence with respect to A .

An HIFS is called *disjoint* if $f_i(F) \cap f_j(F) = \emptyset$ for all $i \neq j$ and $1 \leq i, j \leq n$. In [2], Ellis and Branton proved the following theorem.

Theorem 1.1. *Let F be the attractor of a disjoint HIFS $(\mathcal{K}; f_1, \dots, f_n)$, and let A be a primitive $(0,1)$ -matrix. Suppose that*

$$s_i d(x, y) \leq d(f_i(x), f_i(y)) \leq \bar{s}_i d(x, y),$$

for all $x, y \in \mathcal{K}$, $1 \leq i \leq n$, and for some constants $0 < s_i \leq \bar{s}_i < 1$. Then, $\dim_H(F_A) \leq u$, where $\rho(A\bar{S}^u) = 1$ and \bar{S} is the diagonal matrix with $\text{diag}(\bar{s}_1, \dots, \bar{s}_n)$.

In the same paper [2], Ellis and Branton made the following conjecture for the lower bound: $\dim_H(F_A) \geq l$ where $\rho(AS^l) = 1$ and S is a diagonal matrix with s_1, \dots, s_n on the diagonal and zeros elsewhere.

An $n \times n$ matrix A is called *irreducible* if for all $1 \leq i, j \leq n$, there exists some finite sequence $(i_l)_{1 \leq l \leq m}$ with $i = i_1$ and $j = i_m$ such that $(A)_{i_l, i_{l+1}}(A)_{i_{l+1}, i_{l+2}} > 0$ for $1 \leq l \leq m$. Every primitive matrix is irreducible, but there exist matrices which are irreducible and not primitive [6]. Let $N \geq 2$ and $\{\mathcal{K}; f_{ij}, (A)_{ij} : 1 \leq i \leq N\}$, where $f_{ij} : \mathcal{K} \rightarrow \mathcal{K}$ is a hyperbolic map for $1 \leq i, j \leq N$ and A is an irreducible $(0,1)$ -matrix. The system $\{\mathcal{K}; f_{ij}, (A)_{ij} : 1 \leq i \leq N\}$ is called a *hyperbolic recurrent IFS*.

A particular case of Roychowdhury's result below not only proves the conjecture proposed by Ellis and Branton, but also generalizes Theorem 1.1 by allowing the matrix A to be irreducible and requiring the IFS to satisfy the OSC:

Theorem 1.2. *Let $\{\mathcal{K}; f_{ij}, (A)_{ij} : 1 \leq i, j \leq N\}$ be a hyperbolic recurrent IFS which satisfies the open set condition and assume A is irreducible. Let F_A be the attractor of the system. Then,*

$$h \leq \dim_H(F_A) \leq H \text{ and } h \leq \overline{\dim}_H(F_A) \leq H,$$

where h and H are given by $\rho(((A)_{ij} s_{ij}^h)_{1 \leq i, j \leq N}) = 1$ and $\rho(((A)_{ij} \bar{s}_{ij})_{1 \leq i, j \leq N}) = 1$ and s_{ij}, \bar{s}_{ij} are given by $s_{ij} d(x, y) \leq d(f_{ij}(x), f_{ij}(y)) \leq \bar{s}_{ij} d(x, y)$.

Although it was not stated so, an attractor described above in Theorems 1.1 and 1.2 can be associated with an SFT defined by a list of forbidden words, each of length 2. Theorem A generalizes Theorem 1.1 completely in \mathbb{R}^n and partially generalizes Theorem 1.2 by allowing the subfractal to be associated with any SFT, regardless of the length of the words in the forbidden word list. Furthermore, we extend the results to subfractals induced by a sofic subshift, which is a broader class than SFTs and, to our knowledge, is new. In the case of Hausdorff dimension, we

remove the requirement that the associated matrices must be irreducible, and hence our results include even more subfractals induced by SFTs and sofic subshifts.

2. BASIC DEFINITIONS AND BACKGROUND

Let $\mathcal{K} \subset \mathbb{R}^n$ be a compact subset and $E \subseteq \mathcal{K}$. Letting $\overline{\mathcal{H}}_\varepsilon^s(E) = \inf_{\mathcal{U} \in \mathcal{O}} \sum_{U \in \mathcal{U}} (\text{diam}(U))^s$,

where \mathcal{O} is the collection of all open ε -covers of E and $s \geq 0$, the s -dimensional Hausdorff outer measure is defined to be $\overline{\mathcal{H}}^s = \lim_{\varepsilon \rightarrow 0} \overline{\mathcal{H}}_\varepsilon^s$. Restricting the outer measure to measurable sets, one defines the s -dimensional Hausdorff measure, H^s . The *Hausdorff dimension* of E , denoted $\dim_H(E)$, is defined as the unique value of s such that:

$$H^r(E) = \begin{cases} 0, & r > s \\ \infty, & r < s. \end{cases}$$

If $N_r(E)$ denotes the smallest number of sets of diameter r that can cover E , the *lower and upper box dimensions* of E are defined, respectively, as [3]:

$$\underline{\dim}_B(E) = \liminf_{r \rightarrow 0} \frac{\log N_r(E)}{-\log r} \text{ and } \overline{\dim}_B(E) = \limsup_{r \rightarrow 0} \frac{\log N_r(E)}{-\log r}.$$

The following relationship between the fractal dimensions defined above are well-known [3]:

$$\dim_H(E) \leq \underline{\dim}_B(E) \leq \overline{\dim}_B(E).$$

Let $\mathcal{A} = \{1, \dots, m\}$ be a finite alphabet. Let Ω_n denote the collection of all words on \mathcal{A} of length n and $\Omega_* = \bigcup_{k=1}^{\infty} \Omega_k$ denote the collection of all finite words of any finite length. Let X denote the compact metric space of all infinite sequences on \mathcal{A} , equipped with the metric d_X defined by $d_X(\omega, \tau) = \frac{1}{2^k}$ where $k = \min\{i : \omega_i \neq \tau_i\}$, for all $\omega = \omega_1\omega_2\dots, \tau = \tau_1\tau_2\dots \in X$. For $\omega \in \Omega_*$, let $\ell(\omega)$ denote the length of the word ω . Let $\sigma : X \rightarrow X$ denote the shift map defined by $\sigma(\omega_1\omega_2\dots) = \omega_2\omega_3\dots$ for all $\omega = \omega_1\omega_2\dots \in X$. We will also adopt the following notations:

$$\omega\tau = \omega_1\dots\omega_n\tau_1\dots\tau_m \text{ for } \omega \in \Omega_n, \tau \in \Omega_m,$$

$$\omega^- = \omega_1\dots\omega_{n-1} \text{ for } \omega \in \Omega_n,$$

$$\omega|_n = \omega_1\dots\omega_n \text{ for all } \omega \in X.$$

We will begin by focusing on specific subshifts, namely, subshifts of finite type (SFTs). An SFT, say Y , is defined by a finite list of forbidden words of finite length. A word $\tau \in \Omega_n$ is *forbidden* if it appears nowhere in ω for all $\omega \in Y$. Any word that is not forbidden is called an *allowable* word. Observe that for any list of forbidden words $F = \{x_1, \dots, x_k\}$, $x_i \in \Omega_*$ for $1 \leq i \leq k$, there exists an integer N such that F can be rewritten as $F = \{y_1, \dots, y_l\}$ where $y_i \in \Omega_N$ for $1 \leq i \leq l$. For more information on SFTs, refer to [6].

Let $\omega = \omega_1\dots\omega_{k-1}, \xi = \xi_1\dots\xi_{k-1} \in \Omega_{k-1}$. We say ω is *compatible* with ξ if $\omega_2\dots\omega_{k-1} = \xi_1\dots\xi_{k-2}$. A compatible pair is a pair $(\omega, \xi) \in \Omega_{k-1} \times \Omega_{k-1}$, where ω is compatible with ξ . Let $(\Omega_{k-1} \times \Omega_{k-1})_{\text{comp}}$ denote the collection of all compatible pairs $(\omega, \xi) \in \Omega_{k-1} \times \Omega_{k-1}$. Define an operation $*$: $(\Omega_{k-1} \times \Omega_{k-1})_{\text{comp}} \rightarrow \Omega_k$ by $\omega * \xi = \omega_1\omega_2\dots\omega_{k-1}\xi_{k-1} = \omega_1\xi_1\xi_2\dots\xi_{k-1}$.

Let X_F be a SFT with forbidden words $F = \{\tau_1, \dots, \tau_l\}$. Without loss of generality, we can assume $\tau_i \in \Omega_k$ for all $1 \leq i \leq l$. Let $W_n(X_F)$ denote all allowable words of length n from X_F for $n \geq 1$. If the subshift X_F is clearly understood in context, we will typically write W_n . Let $W_* = \bigcup_{k=1}^{\infty} W_k$ denote the collection of all finite allowable strings.

Let $N = m^{k-1}$, where $m = |\mathcal{A}|$ and $\ell(\tau_i) = k$ for all $\tau_i \in F$. We will construct an $N \times N$ adjacency matrix A as follows. Label the rows with all possible words (both allowable and forbidden) of length $k-1$, i.e. label the rows with $\{\omega_1, \dots, \omega_N\} = \Omega_{k-1}$. Label the corresponding columns similarly. Let the entry be $a_{ij} = 0$ if ω_i is not compatible with ω_j and $a_{ij} = 0$ if ω_i is compatible with ω_j but $\omega_i * \omega_j \in F$. The entry $a_{ij} = 1$ if ω_i is compatible with ω_j and $\omega_i * \omega_j$ is an allowable word.

For the sake of clarity, consider the following examples. First, consider the SFT on the alphabet $\mathcal{A} = \{1, 2\}$ with forbidden word $F_1 = \{22\}$. The associated matrix will be of the form:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Next, let us consider a SFT on the same alphabet $\mathcal{A} = \{1, 2\}$ but with forbidden word list $F_2 = \{112, 211, 222\}$. Since each forbidden word has length 3, we will need to consider a 4×4 matrix since $|\Omega_2| = 4$. We will choose the following labeling of rows: $R_1 \rightarrow 11, R_2 \rightarrow 12, R_3 \rightarrow 21, R_4 \rightarrow 22$. The corresponding matrix will be of the form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Here, the entries $a_{12} = a_{31} = a_{44} = 0$ correspond to the forbidden words 112, 211, 222, respectively. The entries $a_{13} = a_{14} = a_{21} = a_{22} = a_{33} = a_{34} = a_{41} = a_{42} = 0$ correspond to pairs which are not compatible. The 1's in the matrix all correspond to compatible pairs which are also allowable words. We will use either X_A or X_F to denote the SFT.

To each such $N \times N$ adjacency matrix, we can associate a directed graph $G_A = (V_A, E_A)$ where $V = \{v_1, v_2, \dots, v_N\}$ and $E = \{e_{i,j}\}_{i,j=1}^N$ where $e_{i,j}$ is an edge from v_i to v_j if the entry $a_{ij} = 1$ from A . A directed graph $G = (V, E)$ is called *strongly connected* if for any two vertices $v_i, v_j \in V$, there exists a path from v_i to v_j .

Proposition 2.1. *A matrix A is irreducible iff it is associated with a graph G_A which is strongly connected.*

For details on Proposition 2.1, see [6]. By the Perron-Frobenius Theorem, we know that if A is an irreducible matrix, then A has a positive eigenvector \mathbf{v}_A corresponding to a positive eigenvalue $\lambda_A \in \mathbb{R}$ such that $|\mu| \leq \lambda_A$ where μ is any eigenvalue of A [6]. For any non-negative $m \times m$ matrix A with a positive eigenvector and corresponding positive eigenvalue λ , there exist constants $b_0, d_0 > 0$

such that

$$b_0 \lambda^n \leq \sum_{i,j=1}^m (A^n)_{ij} \leq d_0 \lambda^n.$$

3. SUBFRACTALS ASSOCIATED WITH A SUBSHIFT

Let $\{\mathcal{K}; f_1, \dots, f_m\}$ be the system defined in the statement of the main theorem, and let \mathcal{F} denote the attractor of the HIFS. If $\mathcal{A} = \{1, \dots, m\}$, where each letter i corresponds to the map f_i for $1 \leq i \leq m$, and $\omega = \omega_1 \dots \omega_n \in \Omega_n$, we will use the following notation:

$$\begin{aligned} f_\omega &= f_{\omega_n} \circ f_{\omega_{n-1}} \circ \dots \circ f_{\omega_1} \\ c_\omega &= c_{\omega_1} c_{\omega_2} \dots c_{\omega_n}. \end{aligned}$$

Define the associated coding map $\pi : X \rightarrow \mathcal{F}$ by $\pi(\omega) = \lim_{n \rightarrow \infty} f_{\omega|n}(\mathcal{K})$.

For each such IFS, we can define a subfractal of \mathcal{F} by only considering the points associated with a word from a subshift. Let X_F be a SFT and define $\mathcal{F}_{X_F} = \{\pi(\omega) : \omega \in X_F\}$.

As defined in section 2, fix an $N \times N$ adjacency matrix. Let $\Omega_{k-1} = \{\tau^1, \tau^2, \dots, \tau^N\}$, $N = m^{k-1}$. Define two other $N \times N$ matrices, S_0 and S , as follows:

$$S_0 = \begin{bmatrix} c_{\tau^1} & 0 & \dots & 0 \\ 0 & c_{\tau^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{\tau^N} \end{bmatrix} \text{ and } S = \begin{bmatrix} c_{i_1} & 0 & \dots & 0 \\ 0 & c_{i_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{i_N} \end{bmatrix},$$

where $i_j \in \mathcal{A}$ for all $1 \leq j \leq N$ and the order of the i_j 's is chosen so that

$$\sum_{i,j=1}^N (S_0 A_0 S)_{i,j} = \sum_{\omega \in \Omega_{k-1}} c_\omega,$$

with adjacency matrix A_0 associated with the full shift. Similarly, we define

$$\bar{S}_0 = \begin{bmatrix} \bar{c}_{\tau^1} & 0 & \dots & 0 \\ 0 & \bar{c}_{\tau^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{c}_{\tau^N} \end{bmatrix} \text{ and } \bar{S} = \begin{bmatrix} \bar{c}_{i_1} & 0 & \dots & 0 \\ 0 & \bar{c}_{i_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \bar{c}_{i_N} \end{bmatrix}.$$

For $t \in \mathbb{R}$, define

$$S^{(t)} = \begin{bmatrix} c_{i_1}^t & 0 & \dots & 0 \\ 0 & c_{i_2}^t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_{i_N}^t \end{bmatrix},$$

and define $S_0^{(t)}$, $\bar{S}^{(t)}$, and $\bar{S}_0^{(t)}$ similarly.

Next, we will define a topological pressure function for calculating bounds for the fractal dimensions. Topological pressure functions have been used to find bounds for fractal dimensions of different types of fractal classes [8].

Definition 3.1. Let X_A be a subshift. The lower topological pressure function of \mathcal{F}_{X_A} is given by $P(t) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\omega \in W_n} c_\omega^t \right)$. Similarly, we define the upper topological pressure function by $\bar{P}(t) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\omega \in W_n} \bar{c}_\omega^t \right)$.

Proposition 3.2. The lower and upper topological pressure functions $P(t)$ and $\bar{P}(t)$ are strictly decreasing, convex, and continuous on \mathbb{R} .

Proof. We will show the proof for $P(t)$. The proof for $\bar{P}(t)$ follows similarly. Let $\delta > 0$. By using the fact that $c_\omega \leq c_{max}^n$ for all $\omega \in W_n$, where $c_{max} = \max_{1 \leq i \leq m} \{c_i\}$, we have:

$$\begin{aligned} P(t + \delta) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\omega \in W_n} c_\omega^{t+\delta} \right) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\omega \in W_n} c_\omega^t c_{max}^{n\delta} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(c_{max}^{n\delta} \sum_{\omega \in W_n} c_\omega^t \right) = \lim_{n \rightarrow \infty} \frac{1}{n} [n\delta \log(c_{max})] + P(t) \\ &= \delta \log(c_{max}) + P(t) < P(t), \text{ since } 0 < c_{max} < 1. \end{aligned}$$

Hence, $P(t)$ is strictly decreasing. If $t_1, t_2 \in \mathbb{R}$ and $a_1, a_2 > 0$ with $a_1 + a_2 = 1$, then, by Hölder's inequality, we have

$$\begin{aligned} P(a_1 t_1 + a_2 t_2) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\omega \in W_n} (c_\omega)^{a_1 t_1 + a_2 t_2} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left[\sum_{\omega \in W_n} ((c_\omega)^{t_1})^{a_1} ((c_\omega)^{t_2})^{a_2} \right] \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \left[\sum_{\omega \in W_n} (c_\omega)^{t_1} \right]^{a_1} \left[\sum_{\omega \in W_n} (c_\omega)^{t_2} \right]^{a_2} \\ &= a_1 P(t_1) + a_2 P(t_2). \end{aligned}$$

Hence, $P(t)$ is a convex function and strictly decreasing, and thus must be continuous. □

Proposition 3.3. There is a unique value $h \in [0, \infty)$ such that $P(h) = 0$.

Proof. If $t = 0$,

$$P(0) = \lim_{n \rightarrow \infty} \log \left(\sum_{\omega \in W_n} c_\omega^0 \right) = \lim_{n \rightarrow \infty} \log(|W_n|) \geq 0.$$

Next, we will look at the case where $t \rightarrow \infty$.

$$\begin{aligned}
P(t) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\omega \in W_n} c_\omega^t \right) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\omega \in W_n} c_{max}^{nt} \right) \\
&= t \log(c_{max}) + \lim_{n \rightarrow \infty} \frac{1}{n} \log(|W_n|) \leq t \log(c_{max}) + \lim_{n \rightarrow \infty} \frac{1}{n} \log(m^n) \\
&= t \log(c_{max}) + \log(m).
\end{aligned}$$

Since $0 < c_{max} < 1$, we must have $[t \log(c_{max}) + \log(m)] \rightarrow -\infty$ as $t \rightarrow \infty$, and hence $\lim_{t \rightarrow \infty} P(t) = -\infty$. By Proposition 3.2, there exists a unique value h such that $P(h) = 0$. \square

Following the same steps as in the proof above, we have:

Proposition 3.4. *There is a unique value $H \in [0, \infty)$ such that $\bar{P}(H) = 0$.*

Proposition 3.5. *Let h and H be the unique values such that $P(h) = 0 = \bar{P}(H)$. Then, $h \leq H$.*

Proof. Assume that $h > H$. Then, $\bar{P}(h) < \bar{P}(H) = 0$. We also know that $c_\omega \leq \bar{c}_\omega$ for all $\omega \in W_n$. Hence,

$$0 = P(h) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\omega \in W_n} c_\omega^h \right) \leq \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\omega \in W_n} \bar{c}_\omega^h \right) = \bar{P}(h) < 0,$$

which is a contradiction. Hence, $h \leq H$. \square

Lemma 3.6. *Let X_A be an SFT associated with matrix A , and let $\{\mathcal{K}; f_i : 1 \leq i \leq m\}$ be an HIFS. Let S_0 and S be matrices associated with the subfractal \mathcal{F}_{X_A} , as above. Then, the associated lower and upper topological pressure functions $P(t)$ and $\bar{P}(t)$ can be written, respectively, as*

$$\begin{aligned}
P(t) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{i,j=1}^N [S_0^{(t)} (AS^{(t)})^{n-k+1}]_{i,j} \right) \text{ and} \\
\bar{P}(t) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{i,j=1}^N [\bar{S}_0^{(t)} (A\bar{S}^{(t)})^{n-k+1}]_{i,j} \right).
\end{aligned}$$

Proof. Recall that if F is a list of forbidden words, all of length k , then A is an $N \times N$ matrix, where $N = |\Omega_{k-1}| = m^{k-1}$. We will prove the assertion by induction. First, the nonzero entries of A correspond to the allowable words of length k . Hence, by definition of A, S_0 , and S , we have

$$\sum_{i,j=1}^N [S_0 AS]_{ij} = \sum_{\omega \in W_k} c_\omega.$$

Now, assume that $\sum_{i,j=1}^N [S_0 (AS)^n]_{ij} = \sum_{\omega \in W_{n+k-1}} c_\omega$ for some $n > 1$. The entries of $S_0 (AS)^n$ consist of sums of contractive factors associated with allowable words of length $n + k - 1$. Now, consider the matrix $S_0 (AS)^n (AS)$. By the definition of A and S , this multiplication will result in entries consisting of sums of contractive

factors associated with allowable words of length $n + k$. Since $S_0(AS)^n$ contains all allowable words of length $n + k - 1$, then we must have $\sum_{i,j=1}^N [S_0(AS)^{n+1}]_{ij} = \sum_{\omega \in W_{n+k}} c_\omega$. Hence,

$$P(t) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\omega \in W_n} c_\omega^t \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{i,j=1}^N [S_0^{(t)}(AS^{(t)})^{n-(k-1)}]_{ij} \right).$$

The proof follows similarly for the upper topological pressure function. \square

4. MAIN THEOREM FOR SFTs

We begin with a technical lemma that will provide bounds needed for the main result.

Lemma 4.1. *Let S_0 , A , and S be defined as in Section 3, where A is an irreducible matrix. Then, for any $t > 0$, there exist positive constants K, L such that*

$$c_{\min}^{(k-1)t} K \lambda_{AS^{(t)}}^n \leq \sum_{i,j=1}^N [S_0^{(t)}(AS^{(t)})^n]_{i,j} \leq c_{\max}^{(k-1)t} L \lambda_{AS^{(t)}}^n,$$

where $c_{\min} = \min_{1 \leq i \leq m} \{c_i\}$, $c_{\max} = \max_{1 \leq i \leq m} \{c_i\}$, $\lambda_{AS^{(t)}}$ is the maximal eigenvalue of $AS^{(t)}$.

Proof. Notice that for every non-zero entry of S_0 , we have $c_{\min}^{k-1} \leq (S_0)_{ij} \leq c_{\max}^{k-1}$, $1 \leq i, j \leq N$. Hence, by the Perron-Frobenius Theorem, we have constants K and L such that

$$\begin{aligned} c_{\min}^{(k-1)t} K \lambda_{AS^{(t)}}^n &\leq c_{\min}^{(k-1)t} \sum_{i,j=1}^N [(AS^{(t)})^n]_{i,j} \leq \sum_{i,j=1}^N [S_0^{(t)}(AS^{(t)})^n]_{i,j} \\ &\leq c_{\max}^{(k-1)t} \sum_{i,j=1}^N [(AS^{(t)})^n]_{i,j} \leq c_{\max}^{(k-1)t} L \lambda_{AS^{(t)}}^n \end{aligned}$$

\square

Remark 2. *By Lemma 4.1, one can show that, for fixed value $t \in [0, \infty]$,*

$$\begin{aligned} P(t) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{i,j=1}^N [S_0^{(t)}(AS^{(t)})^n]_{i,j} \right) \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \log (c_{\max}^{(k-1)t} L \lambda_{AS^{(t)}}^n) = \log(\lambda_{AS^{(t)}}) = \log(\rho(AS^{(t)})), \end{aligned}$$

where $\rho(AS^{(t)})$ denotes the spectral radius of $AS^{(t)}$. Similarly, we can show that $\log(\rho(AS^{(t)})) \leq P(t)$, and hence $P(t) = \log(\rho(AS^{(t)}))$. Therefore, the unique value h such that $P(h) = 0$ is also the value of h such that $\rho(AS^{(h)}) = 1$. Analogously, we can show that the value H such that $\bar{P}(H) = 0$ is also the value of H that satisfies $\rho(A\bar{S}^{(H)}) = 1$.

Proposition 4.2. *Let h be the unique zero of the lower topological pressure function. There exist positive constants K_0, L_0 such that*

$$K_0 \leq \sum_{\omega \in W_n} c_\omega^h \leq L_0,$$

for all $n \geq 1$.

Proof. Let $s < h$. Then, $P(s) > P(h) = 0$. So, we have

$$\begin{aligned} 0 < P(s) &= \lim_{p \rightarrow \infty} \frac{1}{np} \log \left(\sum_{\omega \in W_{np}} c_\omega^s \right) \leq \lim_{p \rightarrow \infty} \frac{1}{np} \log \left(\sum_{\omega \in W_n} c_\omega^s \right)^p \\ &= \frac{1}{n} \log \left(\sum_{\omega \in W_n} c_\omega^s \right). \end{aligned}$$

Hence, $\sum_{\omega \in W_n} c_\omega^s > 1$, and it follows that $\sum_{\omega \in W_n} c_\omega^h \geq 1$.

Now, assume that $s > h$. Then, $0 = P(h) > P(s)$. So, by Lemma 4.1, we have

$$\begin{aligned} 0 > P(s) &= \lim_{p \rightarrow \infty} \frac{1}{np} \log \left(\sum_{\omega \in W_{np}} c_\omega^s \right) = \lim_{p \rightarrow \infty} \frac{1}{np} \log \left(\sum_{i,j=1}^N [S_0^{(s)}(AS^{(s)})^{np}]_{i,j} \right) \\ &\geq \lim_{p \rightarrow \infty} \frac{1}{np} \log \left(c_{min}^{(k-1)s} K \lambda_{AS^{(s)}}^{np} \right) = \frac{1}{n} \log(\lambda_{AS^{(s)}}^n) \\ &\geq \frac{1}{n} \log \left(\frac{1}{L c_{max}^{(k-1)s}} \sum_{i,j=1}^N [S_0^{(s)}(AS^{(s)})^n]_{i,j} \right) = \frac{1}{n} \log \left(\frac{1}{L c_{max}^{(k-1)s}} \sum_{\omega \in W_n} c_\omega^s \right). \end{aligned}$$

Hence, $\sum_{\omega \in W_n} c_\omega^s < L c_{max}^{(k-1)s}$, which implies that $\sum_{\omega \in W_n} c_\omega^h \leq L c_{max}^{(k-1)h}$. \square

Following similar steps in the proof of Proposition 4.2, we have:

Proposition 4.3. *Let H be the unique zero of the upper topological pressure function. There exist positive constants K_1, L_1 such that*

$$K_1 \leq \sum_{\omega \in W_n} \bar{c}_\omega^H \leq L_1.$$

In order to show that h is a lower bound for $\dim_H(\mathcal{F})$, we will utilize the uniform mass distribution principle from Falconer [4]. Hence, we must define an appropriate Borel probability measure to satisfy the principle. Let h be the unique value such that $P(h) = 0$. Let $\omega \in \Omega_n$ and let $[\omega] = \{\tau \in \Omega_\infty : \tau_i = \omega_i, 1 \leq i \leq n\}$ be the cylinder set corresponding to ω . We will use the fact that $c_{\omega\tau} = c_\omega c_\tau$. Define

$$\nu_n([\omega]) = \frac{\sum_{\omega\tau \in W_{n+\ell(\omega)}} c_{\omega\tau}^h}{\sum_{\tau \in W_{n+\ell(\omega)}} c_\tau^h}.$$

For all $n \geq 1$ and any $\omega \in W_*$, we have by Proposition 4.2,

$$0 \leq \frac{\sum_{\omega\tau \in W_{n+\ell(\omega)}} c_{\omega\tau}^h}{L_0} \leq \nu_n([\omega]) \leq \frac{c_\omega^h \sum_{\tau \in W_n} c_\tau^h}{\sum_{\tau \in W_{n+\ell(\omega)}} c_\tau^h} \leq \frac{L_0}{K_0} c_\omega^h < \infty.$$

Hence, for all $\omega \in W_*$, $\text{Lim}_{n \rightarrow \infty} \nu_n(\llbracket \omega \rrbracket)$ exists, where Lim denotes the Banach limit. Let $\nu(\llbracket \omega \rrbracket) = \text{Lim}_{n \rightarrow \infty} \nu_n(\llbracket \omega \rrbracket)$.

Also, notice that

$$\begin{aligned} \sum_{i=1}^m \nu(\llbracket \omega i \rrbracket) &= \text{Lim}_{n \rightarrow \infty} \sum_{i=1}^m \frac{\sum_{\tau \in W_{n+\ell(\omega i)}} c_{\omega i \tau}^h}{\sum_{\tau \in W_{n+\ell(\omega i)}} c_{\tau}^h} \\ &= \text{Lim}_{n \rightarrow \infty} \frac{\sum_{\tau \in W_{n+1+\ell(\omega)}} c_{\omega \tau}^h}{\sum_{\tau \in W_{n+1+\ell(\omega)}} c_{\tau}^h} = \nu(\llbracket \omega \rrbracket). \end{aligned}$$

Hence, by applying Kolmogorov extension theorem, we can extend ν to a unique Borel probability measure γ on X_A . Let $\mu_h = \gamma \circ \pi^{-1}$, where π is the coding map. Hence, μ_h is supported on \mathcal{F}_{X_A} .

Corollary 4.4. *There exist constants $K_0, L_0 > 0$ such that*

$$\mu_h(f_\omega(\mathcal{K})) \leq \frac{L_0}{K_0} c_\omega^h.$$

Proof. By definition of μ_h and Proposition 4.2, we have

$$\mu_h(f_\omega(\mathcal{K})) = \nu(\llbracket \omega \rrbracket) = \frac{\sum_{\tau \in W_n} c_{\omega \tau}^h}{\sum_{\tau \in W_{n+\ell(\omega)}} c_{\tau}^h} \leq \frac{c_\omega^h \sum_{\tau \in W_n} c_{\tau}^h}{\sum_{\tau \in W_{n+|\omega|}} c_{\tau}^h} \leq c_\omega^h \frac{L_0}{K_0}.$$

□

Proposition 4.5. *For $0 < r < 1$ and $x \in \mathcal{F}$, the ball $B(x, r)$ intersects at most M elements of $\mathcal{U}_r = \{f_\omega(\mathcal{K}) : |f_\omega(\mathcal{K})| \leq r < |f_{\omega^-}(\mathcal{K})|\}$, where M is finite and independent of r .*

Proof. Let $0 < r < 1$ and $x \in \mathcal{F}$. Let $W_r = \{\omega \in W_* : f_\omega(\mathcal{K}) \cap B(x, r) \neq \emptyset, f_\omega(\mathcal{K}) \in \mathcal{U}_r\}$ and $|W_r| = M$. Let $y \in B(x, r)$ and $z \in f_\omega(\mathcal{K})$ where $\omega \in W_r$. Notice that

$$d(y, z) \leq |B(x, r)| + |f_\omega(\mathcal{K})| \leq 3r.$$

Hence, $\{f_\omega(\mathcal{K}) : \omega \in W_r\} \subset B(x, 3r)$. For any $f_\omega(\mathcal{K}) \in \mathcal{U}_r$, we have

$$|f_\omega(\mathcal{K})| \geq c_{\min} |f_{\omega^-}(\mathcal{K})| > c_{\min} r.$$

Due to the open set condition, there exists a ball B_a of radius $a > 0$ such that $B_a \subset \mathcal{K}$ and $f_\omega(B_a) \cap f_\tau(B_a) = \emptyset$ for $\omega, \tau \in W_r$. For each $\omega \in W_r$, we have $f_\omega(B_a) \subset f_\omega(\mathcal{K})$. Let m denote Lebesgue measure on \mathcal{K} . Since the balls are disjoint and contained in $B(x, 3r)$, we have

$$\sum_{\omega \in W_r} m(f_\omega(B_a)) \leq m(B(x, 3r)).$$

Using the fact that $|f_\omega(\mathcal{K})| > c_{\min} r$, we have

$$M \cdot m(B(x, ac_{\min} r)) \leq m(B(x, 3r)).$$

Hence, $M \leq \frac{m(B(x, 3r))}{m(B(x, ac_{\min} r))}$. Since the ratio compares concentric balls, each with

a radius equal to a constant multiple of r , we can let $M \leq \left\lceil \frac{m(B(x, 3r))}{m(B(x, c_{\min} r))} \right\rceil < \infty$, which satisfies the assertion of the proposition.

□

Theorem 4.6. *Let h, H be the unique values such that $P(h) = 0 = \bar{P}(H)$. Then, $h \leq \dim_H(\mathcal{F}) \leq H$.*

Proof. Let $\mathcal{U}_n = \{f_\omega(\mathcal{K}) : \omega \in W_n\}$. Notice that \mathcal{U}_n is a cover for all $n \geq 1$. Hence, by Proposition 4.3, we have

$$\begin{aligned} \mathcal{H}^H(\mathcal{F}) &= \liminf_{\varepsilon \rightarrow 0} \sum_{E \in \mathcal{E}} |E|^H \leq \lim_{n \rightarrow \infty} \sum_{\omega \in W_n} |f_\omega(\mathcal{K})|^H \\ &\leq \lim_{n \rightarrow \infty} \sum_{\omega \in W_n} |\mathcal{K}|^H \bar{c}_\omega^H \leq |\mathcal{K}|^H \cdot L_1 < \infty. \end{aligned}$$

Thus, $\dim_h(\mathcal{F}) \leq H$. Let $r > 0$ and $B(x, r)$ be a ball centered at $x \in \mathcal{F}$. By Proposition 4.5, $B(x, r)$ intersects at most M elements of the cover \mathcal{U}_r . Let \mathcal{U}_M denote the subset of \mathcal{U}_r consisting of all elements that intersect $B(x, r)$ and W_M denote all allowable words associated with an element of \mathcal{U}_M . By Corollary 4.4, we have

$$\begin{aligned} \frac{\mu_h(B(x, r))}{r^h} &\leq \frac{\sum_{f_\omega(\mathcal{K}) \in \mathcal{U}_M} \mu_h(f_\omega(\mathcal{K}))}{r^h} \leq \frac{\sum_{\omega \in W_M} \frac{L_0}{K_0} c_\omega^h}{r^h} \\ &\leq \frac{M \frac{L_0}{K_0} |\mathcal{K}|^{-h} r^h}{r^h} = M \frac{L_0}{K_0} |\mathcal{K}|^{-h}. \end{aligned}$$

Hence, $\limsup_{r \rightarrow 0} \frac{\mu_h(B(x, r))}{r^h} \leq M \frac{L_0}{K_0} |\mathcal{K}|^{-h} < \infty$. By the uniform mass distribution principle [4], we have $\mathcal{H}^h(\mathcal{F}) \geq \frac{M \frac{L_0}{K_0} |\mathcal{K}|^{-h}}{\mu_h(\mathcal{F})} > 0$. Thus, $\dim_H(\mathcal{F}) \geq h$. \square

Theorem 4.7. *Let h, H be the unique values such that $P(h) = 0 = \bar{P}(H)$. Then, $h \leq \underline{\dim}_B(\mathcal{F}) \leq H$.*

Proof. The following relationship between Hausdorff and box dimensions is well-known:

$$\dim_H(\mathcal{F}) \leq \underline{\dim}_B(\mathcal{F}) \leq \overline{\dim}_B(\mathcal{F}).$$

Hence, it suffices to show that $\underline{\dim}_B(\mathcal{F}) \leq H$. Let $\mathcal{U}_r = \{f_\omega(\mathcal{K}) : |f_\omega(\mathcal{K})| \leq r < |f_{\omega^-}(\mathcal{K})|\}$, $k = \min\{|\omega| : f_\omega(\mathcal{K}) \in \mathcal{U}_r\}$, and $\mathcal{O}_k = \{f_\omega(\mathcal{K}) : \omega \in W_k\}$. Notice that $\bigcup_{f_\omega(\mathcal{K}) \in \mathcal{U}_r} f_\omega(\mathcal{K}) \subseteq \bigcup_{f_\omega(\mathcal{K}) \in \mathcal{O}_k} f_\omega(\mathcal{K})$. Hence, by Proposition 4.4, we have

$$\sum_{f_\omega(\mathcal{K}) \in \mathcal{U}_r} |f_\omega(\mathcal{K})|^H \leq \sum_{f_\omega(\mathcal{K}) \in \mathcal{O}_k} |f_\omega(\mathcal{K})|^H \leq |\mathcal{K}|^H \sum_{\omega \in W_k} \bar{c}_\omega^H \leq |\mathcal{K}|^H L_1.$$

Also, for $f_\omega(\mathcal{K}) \in \mathcal{U}_r$,

$$|f_\omega(\mathcal{K})| \geq |f_{\omega^-}(\mathcal{K})| \cdot c_{\min} > r c_{\min}.$$

Let $N_r(\mathcal{F})$ denote the smallest number of sets of diameter at most r which form a cover of \mathcal{F} . Then,

$$(r c_{\min})^H N_r(\mathcal{F}) \leq |f_\omega(\mathcal{K})|^H N_r(\mathcal{F}) \leq \sum_{f_\omega(\mathcal{K}) \in \mathcal{U}_r} |f_\omega(\mathcal{K})|^H \leq |\mathcal{K}|^H L_1.$$

Hence, $N_r(\mathcal{F}) \leq (r c_{\min})^{-H} |\mathcal{K}|^H L_1$, and thus

$$\frac{\log(N_r(\mathcal{F}))}{-\log(r)} \leq \frac{\log(L_1 |\mathcal{K}|^H) - H \log(r c_{\min})}{-\log(r)} = \frac{\log(L_1 |\mathcal{K}|^H)}{-\log(r)} + \frac{H \log(c_{\min})}{\log(r)} + H.$$

By the definition of upper box dimension, we have

$$\overline{\dim}_B(\mathcal{F}) = \limsup_{r \rightarrow 0} \frac{\log(N_r(\mathcal{F}))}{-\log(r)} \leq \limsup_{r \rightarrow 0} \left[\frac{\log(L_1 |\mathcal{K}|^H)}{-\log(r)} + \frac{H \log(c_{\min})}{\log(r)} \right] + H = H.$$

□

Remark 3. For $E \subset \mathcal{K}$, the following inequalities are well-known:

$$\dim_H(E) \leq \dim_P(E) \leq \overline{\dim}_B(E) \text{ and } \dim_H(E) \leq \underline{\dim}_B(E) \leq \overline{\dim}_B(E),$$

where $\dim_P(E)$ denote the packing dimension of E . For more information on packing dimension, see [1]. Hence, we have also shown that

$$h \leq \dim_P(\mathcal{F}) \leq H \text{ and } h \leq \underline{\dim}_B(\mathcal{F}) \leq H.$$

5. MAIN THEOREM FOR SOFIC SUBSHIFTS

In this section, we will extend the assertions from Theorems 4.6 and 4.7 to sofic subshifts. Recall that SFTs are sofic subshifts, but there exist sofic subshifts which cannot be represented as a SFT. A common characterization of a sofic shift (Y, σ) is that it must be a factor of some SFT, some (X, σ) . That is, there exists a continuous map $\psi : X \rightarrow Y$ such that $\sigma \circ \psi = \psi \circ \sigma$.

We adopt the following definitions from [6]. Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph, consisting of a graph G with edge set \mathcal{E} and a labeling $\mathcal{L} : \mathcal{E} \rightarrow \mathcal{A}$, where \mathcal{A} is the finite alphabet. A subset X of the full shift is a *sofic subshift* if $X = X_{\mathcal{G}}$ for some labeled graph \mathcal{G} . Let $\mathcal{G} = (G, \mathcal{L})$ be a labeled graph. \mathcal{G} is called *right-resolving* if for each vertex v in G , all edges leaving v have different labels.

It is known that every sofic shift has a right-resolving graph presentation [6]. Hence, if $X_{\mathcal{G}}$ is a sofic subshift, we will assume that \mathcal{G} is a right-resolving presentation. Notice that \mathcal{G} has k states, which correspond to k total vertices from the graph G . Now, define a $k \times k$ adjacency matrix $M_{\mathcal{G}}$ by defining the entries as $m_{i,j} = \sum_{e_{i,j}} \mathcal{L}(e_{i,j})$, where $e_{i,j}$ is an edge from vertex v_i to v_j in the graph G . For more information on sofic subshifts and associated graphs, see [6]. We define another $k \times k$ matrix $M_{\mathcal{G},t}$ by defining the entries as $m_{i,j}^{(t)} = \sum_{e_{i,j}} \mathcal{L}(e_{i,j})^t$.

Let $\{\mathcal{K}; f_1, \dots, f_m\}$ be a hyperbolic IFS with $c_i d(x, y) \leq d(f_i(x), f_i(y)) \leq \bar{c}_i d(x, y)$ for $1 \leq i \leq m$ and all $x, y \in \mathcal{K}$. We will define two $k \times k$ matrices, $A_{\mathcal{G},t}$ and $\bar{A}_{\mathcal{G},t}$ similar to the matrix $M_{\mathcal{G},t}$ above. Let $A_{\mathcal{G},t}$ be defined by the entries $a_{i,j}^{(t)} = \sum_{e_{i,j}} c_{(\mathcal{L}(e_{i,j}))}^t$, where $a_{i,j}^{(t)}$ denotes the (i, j) -th entry of $A_{\mathcal{G},t}$. Let $\bar{A}_{\mathcal{G},t}$ be defined by the entries $\bar{a}_{i,j}^{(t)} = \sum_{e_{i,j}} \bar{c}_{(\mathcal{L}(e_{i,j}))}^t$.

Lemma 5.1. Let $X_{\mathcal{G}}$ be a sofic subshift, where $\mathcal{G} = (G, \mathcal{L})$. If G has k vertices, then

$$\frac{1}{k} \sum_{i,j=1}^k [A_{\mathcal{G},t}^n]_{i,j} \leq \sum_{\omega \in W_n} c_{\omega}^t \leq \sum_{i,j=1}^k [A_{\mathcal{G},t}^n]_{i,j},$$

where W_n denotes all allowable words of length n from $X_{\mathcal{G}}$.

Proof. Let $\omega \in W_n$ for some $n \geq 1$. Notice that there may be more than one representation for ω in \mathcal{G} . Since $\sum_{i,j=1}^k [A_{\mathcal{G}}^n]_{i,j}$ sums contractive factors related to all

labeled paths of length n in \mathcal{G} , then $\sum_{\omega \in W_n} c_\omega^t \leq \sum_{i,j=1}^k [A_{\mathcal{G},t}^n]_{ij}$. Now, if G has k vertices, then \mathcal{G} also has k vertices. By assumption, \mathcal{G} is right-resolving, meaning no two edges leaving the same vertex have the same label. Hence, any $\omega \in W_n$ can have at most k representations in \mathcal{G} . Therefore, for fixed value k , $\frac{1}{k} \sum_{i,j=1}^k [A_{\mathcal{G},t}^n]_{ij} \leq \sum_{\omega \in W_n} c_\omega^t$. \square

Theorem 5.2. *Let $\{\mathcal{K}; f_1, \dots, f_m\}$ be a hyperbolic IFS with $c_i d(x, y) \leq d(f_i(x), f_i(y)) \leq \bar{c}_i d(x, y)$ for $1 \leq i \leq m$ and all $x, y \in \mathcal{K}$. Let $X_{\mathcal{G}}$ be a sofic subshift on the alphabet $\mathcal{A} = \{1, \dots, m\}$ and $\mathcal{F}_{\mathcal{G}}$ be the subfractal defined by the IFS and $X_{\mathcal{G}}$. Suppose $A_{\mathcal{G}}$ is irreducible. If $\rho(A_{\mathcal{G},h}) = 1 = \rho(\bar{A}_{\mathcal{G},H})$, then*

$$h \leq \dim_H(\mathcal{F}_{\mathcal{G}}) \leq H \text{ and } h \leq \overline{\dim}_B(\mathcal{F}_{\mathcal{G}}) \leq H.$$

Proof. By Lemma 5.1, we can rewrite the topological lower and upper pressure function as

$$P(t) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\omega \in W_n} c_\omega^t \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{i,j=1}^k [A_{\mathcal{G},t}^n]_{ij} \right) \text{ and}$$

$$\bar{P}(t) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{i,j=1}^k [\bar{A}_{\mathcal{G},t}^n]_{ij} \right).$$

The remainder of the proof follows as in Theorem 4.6 and Theorem 4.7. \square

Remark 4. *Similar to Remark 2, the values of h and H such that $P(h) = 0 = \bar{P}(H)$ also satisfy $\rho(A_{\mathcal{G},h}) = 1 = \rho(\bar{A}_{\mathcal{G},H})$.*

Remark 5. *Similar to Remark 3, due to known relationships between Hausdorff, packing, upper and lower box dimensions, we also have*

$$h \leq \dim_P(\mathcal{F}_{\mathcal{G}}) \leq H \text{ and } h \leq \underline{\dim}_B(\mathcal{F}_{\mathcal{G}}) \leq H.$$

6. GENERALIZATION TO REDUCIBLE MATRICES

In this section, we will eliminate the irreducibility condition on the matrices in the case of Hausdorff dimension. Consider the case where $A_{\mathcal{G}}$ (or A_G if we have an SFT) is a reducible matrix. Let A be a reducible $m \times m$ $(0,1)$ -matrix, and \mathcal{G} be the associated graph. Since A is a reducible matrix, the graph \mathcal{G} is not strongly connected, but it contains a finite number of strongly connected components, say C_1, \dots, C_k . To each component, we can associate a submatrix A_1, \dots, A_k where A_i is irreducible for $1 \leq i \leq k$. Now, we can simultaneously permute the rows and columns of A to obtain:

$$\tilde{A} = \begin{bmatrix} A_k & 0 & 0 & \cdots & 0 \\ * & A_{k-1} & 0 & \cdots & 0 \\ * & * & A_{k-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & A_1 \end{bmatrix}.$$

For further details on this process, see [6].

The process used to obtain \tilde{A} from A will not affect the characteristic polynomial, and hence A and \tilde{A} have the same eigenvalues. We can also examine A^n and \tilde{A}^n similarly; that is, by simultaneously interchanging rows and columns of A , each entry of A^n will appear in \tilde{A}^n , although possibly in a different entry position.

Hence, we can assume $\sum_{i,j=1}^m (A^n)_{ij} = \sum_{i,j=1}^m (\tilde{A}^n)_{ij}$ [6]. Without loss of generality, we will assume that A is in the form of \tilde{A} .

If A is a reducible $m \times m$ matrix with irreducible components A_1, \dots, A_k , let A_i be an $m_i \times m_i$ matrix for $1 \leq i \leq k$. For $l < k$, we define the set

$$\text{trn}(A_l, A_p) = \{a_{ij} \neq 0 : \sum_{s=l+1}^k m_s \leq i \leq \sum_{s=l}^k m_s, \sum_{s=p+1}^k m_s \leq j \leq \sum_{s=p}^k m_s\}$$

of all non-zero entries from $A_{\mathcal{G}}$ corresponding to a transitional edge in \mathcal{G} from component C_l associated with A_l to the component C_p associated with A_p . Let W_{trn} denote all finite words in Ω_k corresponding to a transitional edge from the graph \mathcal{G} .

Each strongly connected component of the graph, C_i , corresponds to an irreducible submatrix, A_i , and a subshift X_{A_i} , $1 \leq i \leq k$. For simplicity, we will talk about the construction of words in X_A by using the strongly connected components C_i , $1 \leq i \leq k$ from \mathcal{G} . Given the structure of the entire graph \mathcal{G} and direction of the transitional edges, words in X_A must begin in a component C_i , move through components C_j , and end in component C_l where $1 \leq i \leq j \leq l \leq k$. To formalize this in the subshift setting, we introduce the following notation.

For $1 \leq i < j \leq k$, let

$$X_{A_i} \otimes X_{A_j} = \{\omega \in X_A : \omega = \tau a \xi, \text{ where } \tau \in W_*(A_i), a \in W_{\text{trn}}, \xi \in X_{A_j}\}.$$

Similarly, for $1 \leq i_1 < i_2 < \dots < i_l \leq k$, we define $X_{A_{i_1}} \otimes \dots \otimes X_{A_{i_l}} = \{\omega \in X_A : \omega = \tau_1 a_1 \tau_2 \dots a_{l-1} \xi, \text{ where } \tau_j \in W_*(A_{i_j}), a_j \in W_{\text{trn}} \text{ for } 1 \leq j < l, \xi \in X_{A_{i_l}}\}.$

Lemma 6.1. *If \mathcal{G} has k irreducible components for $k \geq 2$, then*

$$X_{\mathcal{G}} = \left(\bigcup_{i=1}^k X_{A_i} \right) \cup \left(\bigcup_{j=2}^k \bigcup_{i_1, \dots, i_j=1}^k X_{A_{i_1}} \otimes \dots \otimes X_{A_{i_j}} \right),$$

where $i_l < i_{l+1}$ for $1 \leq l < j$.

Proof. We will use induction for this argument. If \mathcal{G} has two strongly connected components, C_1 and C_2 , with at least one transitional edge from C_1 to C_2 then it follows that $X_{A_1} \cup X_{A_2} \cup (X_{A_1} \otimes X_{A_2}) \subseteq X_{A_{\mathcal{G}}}$. Now, let $\omega \in X_{A_{\mathcal{G}}}$. Then, ω must begin in either C_1 , C_2 , or on a transitional edge. If ω starts in C_2 , then $\omega \in X_{A_2}$ because there are no transitional edges leaving C_2 in \mathcal{G} . If ω starts on a transitional edge, then $\omega \in (X_{A_1} \otimes X_{A_2})$ because it is of the form $\omega = \tau * a * \xi$ where τ is the empty word from $W_*(A_1)$. If ω starts in C_1 , then either $\omega \in X_{A_1}$ or $\omega \in (X_{A_1} \otimes X_{A_2})$. Hence, we must have

$$X_{A_1} \cup X_{A_2} \cup (X_{A_1} \otimes X_{A_2}) = X_{A_{\mathcal{G}}}.$$

Now, assume \mathcal{G} is a connected graph with k strongly connected components, and consider the subgraph, say $\mathcal{G}|_{(k-1)}$, consisting of the first $k-1$ components. Assume that

$$X_{\mathcal{G}|_{(k-1)}} = \left(\bigcup_{i=1}^{k-1} X_{A_i} \right) \cup \left(\bigcup_{j=2}^{k-1} \bigcup_{i_1, \dots, i_j=1}^{k-1} X_{A_{i_1}} \otimes \cdots \otimes X_{A_{i_j}} \right).$$

By comparing the graphs \mathcal{G} and $\mathcal{G}|_{(k-1)}$ and their corresponding subshifts $X_{\mathcal{G}}$ and $X_{\mathcal{G}|_{(k-1)}}$, we can conclude that any word in $X_{\mathcal{G}} - X_{\mathcal{G}|_{(k-1)}}$ will end in C_k . Hence,

$$X_{A_{\mathcal{G}}} = X_{A_{\mathcal{G}_{k-1}}} \cup X_{A_k} \cup \left(\bigcup_{i_1, \dots, i_j=1}^{k-1} X_{A_{i_1}} \otimes \cdots \otimes X_{A_{i_j}} \otimes X_{A_k} \right),$$

which satisfies the assertion. \square

Proposition 6.2. *Let $A_{\mathcal{G}}$ be a reducible matrix with irreducible components A_1, \dots, A_k . Then,*

$$h_{i_j} \leq \dim_H(\mathcal{F}_{X_{A_{i_1}} \otimes \cdots \otimes X_{A_{i_j}}}) \leq H_{i_j},$$

where

$$h_{i_j} \leq \dim_H(\mathcal{F}_{X_{A_{i_j}}}) \leq H_{i_j}$$

and h_{i_j}, H_{i_j} are the bounds from Theorem 5.2.

Proof. Consider a finite word $\tau_1 a_1 \tau_2 a_2 \dots \tau_{j-1} a_{j-1}$ where $\tau_l \in W_*(A_{i_l})$ for $1 \leq l \leq j-1$ and $a_l \in \text{trn}(A_{i_l}, A_{i_{l+1}})$. For any $n \geq 1$, there are finitely many words $\tau_l \in W_*(A_{i_l})$ with $\ell(\tau_l) \leq n$, for all $1 \leq l \leq j$. Hence, there are finitely many words of the form $\tau_1 a_1 \dots \tau_{j-1} a_{j-1}$ of length n . So, the collection $S = \{\tau_1 a_1 \dots \tau_{j-1} a_{j-1} : \tau_l \in W_*(A_{i_l}) \text{ for } 1 \leq l \leq j-1, a_l \in \text{trn}(A_{i_l}, A_{i_{l+1}}), \ell(\tau_1 a_1 \dots \tau_{j-1} a_{j-1}) < \infty\}$ is at most countable since $W_*(A_{i_l})$ is countable for $i \leq i_l \leq k$. For $\omega \in S$, let $\omega X_{A_{i_j}} = \{\omega \xi \in X_{A_{\mathcal{G}}} : \xi \in X_{A_{i_j}}\}$. Then, $\dim_H((\mathcal{F}_{X_{A_{i_1}} \otimes \cdots \otimes X_{A_{i_j}}})) = \sup_{\omega \in S} \dim_H(\mathcal{F}_{\omega X_{A_{i_j}}})$.

First, notice that $\mathcal{F}_{\omega X_{A_i}} = \{f_{\omega \xi}(x) : \xi \in X_{A_i} \text{ and } x \in \mathcal{K}\}$ for any $1 \leq i \leq k$. Recall that $f_{\omega \xi}(x) = f_{\xi} \circ f_{\omega}(x)$ and $f_{\omega}(x) \in \mathcal{K}$ for all $x \in \mathcal{K}$. Hence, $\mathcal{F}_{\omega X_{A_i}} \subseteq \mathcal{F}_{\omega A_i}$. Hence,

$$\dim_H(\mathcal{F}_{\omega X_{A_i}}) \leq \dim_H(\mathcal{F}_{X_{A_i}}) \leq H_i,$$

where H_i is the bound from Theorem 5.2.

Let $\omega \in S$ with $\ell(\omega) = m$ and A_i be an irreducible block in A . Consider $\dim_H(\mathcal{F}_{\omega X_{A_i}})$. Although ωX_{A_i} is not necessarily a subshift itself, we can apply similar techniques used to prove Theorem 5.2 to show that the zero of the lower topological pressure function $P_{\omega, i}(t) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\tau \in W_n(\omega X_{A_i})} c_{\tau}^t \right)$ is a lower bound for

$\dim_H(\mathcal{F}_{\omega X_{A_i}})$. Notice that

$$\begin{aligned} P_{\omega,i}(t) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\tau \in W_n(\omega X_{A_i})} c_\tau^t \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\sum_{\tau \in W_{n-m}(X_{A_i})} c_\omega^t c_\tau^t \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\log(c_\omega^t) + \log \left(\sum_{\tau \in W_{n-m}(X_{A_i})} c_\tau^t \right) \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n-m} \log \left(\sum_{\tau \in W_{n-m}(X_{A_i})} c_\tau^t \right) \\ &= P_i(t), \end{aligned}$$

where $P_i(t)$ is the lower topological pressure function associated with the subfractal $\mathcal{F}_{X_{A_i}}$. Hence,

$$h_i \leq \dim_H(\mathcal{F}_{\omega X_{A_i}}).$$

Thus,

$$\dim_H(\mathcal{F}_{X_{A_{i_1 1}} \oplus \dots \oplus X_{A_{i_j}}}) = \sup_{\omega \in S} \dim_H(\mathcal{F}_{\omega X_{A_{i_j}}}) \leq H_{i_j} \text{ and}$$

$$h_{i_j} \leq \dim_H(\mathcal{F}_{X_{A_{i_1 1}} \oplus \dots \oplus X_{A_{i_j}}}),$$

where h_{i_j} and H_{i_j} are the zeros of the upper and lower topological pressure functions $P_{i_j}(t)$ and $\overline{P}_{i_j}(t)$ with respective the subfractal $\mathcal{F}_{X_{A_{i_j}}}$ for some $1 \leq i_j \leq k$. \square

For a similar statement about subshifts with a reducible matrix A , we have, by Lemma 6.1,

$$\mathcal{F}_{X_{A_G}} = \left(\bigcup_{i=1}^k \mathcal{F}_{X_{A_i}} \right) \cup \left(\bigcup_{j=2}^k \bigcup_{1 \leq i_1 < \dots < i_j \leq k} \mathcal{F}_{X_{A_{i_1}} \oplus \dots \oplus X_{A_{i_j}}} \right).$$

Thus, by Proposition 6.2, we have the following theorem.

Theorem 6.3. *Let X_{A_G} be a sofic subshift associated with matrix A_G . Assume A_G has irreducible components A_1, \dots, A_k . Let $\mathcal{F}_{X_{A_G}}$ and $\mathcal{F}_{X_{A_i}}$ denote the sub-fractals associated with the subshifts X_{A_G} and X_{A_i} , respectively. Then,*

$$\max_{1 \leq i \leq k} \{h_i\} \leq \dim_H(\mathcal{F}_{X_{A_G}}) \leq \max_{1 \leq i \leq k} \{H_i\},$$

where $P_i(h_i) = 0 = \overline{P}_i(H_i)$ given in Theorem 5.2 for all $1 \leq i \leq k$.

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